Proof of the Second Partials Test

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1. Proof of the Second Derivative Test from Calc I (using Calc II)

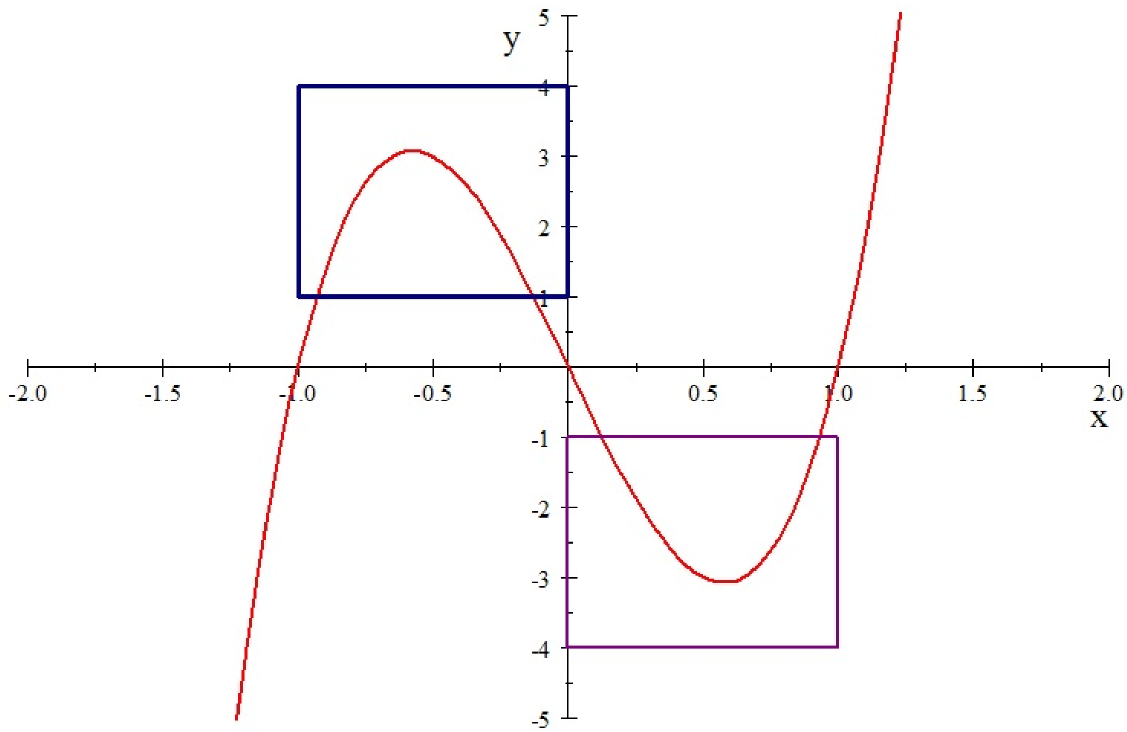
Recall from Calc II that the *Taylor polynomial* of a function f at a point a is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots$$

If a is a critical point of f, that is $f'(a) = 0$, then the Taylor polynomial reads

$$f(x) = f(a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots$$

This means that when x is very close to a (so that the quantity $(x-a)$ is small), the behavior of f is near completely determined by the second order term: $\frac{1}{2}f''(a)(x-a)^2$. This is because $(x-a)^2 >> (x-a)^3$ when $(x-a)$ is very small. So, we see that if $f''(a) > 0$, then near a, f looks like a parabola opening upward, hence a is a local minimum; and if $f''(a) < 0$, then f looks like a parabola opening downward, hence a is a local maximum. Here is a graph of the function $f(x) = 8x(x-1)(x+1)$:



Notice how, inside the blue box (which contains the local maximum of $f(x)$), the function looks almost like (but not exactly like) a parabola that opens downward; and inside the purple box (which contains the local minimum of $f(x)$), the function looks almost like a parabola that opens upward.

Now, if the second derivative is also zero at a, $f''(a) = 0$, but the third derivative is not zero, $f'''(a) \neq 0$, then the Taylor series is dominated by the third order term: $\frac{1}{6}f'''(a)(x-a)^3$. This explains why the second derivative being zero gives a point of inflection. We can continue onward like this for as long as necessary (e.g., if $f'''(a) = 0$, then move on to $f^{(4)}(a)$ so that the behavior of f near a is near completely determined by the fourth order term $\frac{1}{24}f^{(4)}(a)(x-a)^4$, etc...). This means that if $f''(a) = 0$, we don't have enough information to determine what type of critical point a is without taking more derivatives (hence the second derivative test fails).

2. Proof of the Second Partials Test

To prove the second partials test, we are going to try to mimic the above proof in the one variable case.

2.1. Multiplying a vector by a matrix. To make our lives easier, we should think of vectors as columns, e.g., instead of writing the vector $\vec{v} = \langle h, k \rangle$, we write $\vec{v} = \begin{pmatrix} h \\ k \end{pmatrix}$. This will make our dealing with the following proof a bit easier. Suppose we have an 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Then the product $A\vec{v}$ is the vector given by

$$A\vec{v} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} a_{11}h + a_{12}k \\ a_{12}h + a_{22}k \end{pmatrix}$$

2.2. Second order Taylor polynomial of $f(x, y)$. We will gloss over some technicalities here, but they can be found in section 4.1 of Susan Colley's book "Vector Calculus" [1]. The second order Taylor polynomial of a C^2 (continuous second partials) $f(x, y)$ about a point $A = (p, q)$ is given by

$$f(x,y) = f(p,q) + \nabla f(p,q) \cdot (\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x} - \vec{a}) \cdot (Hf(p,q)(\vec{x} - \vec{a})) + R_2(\vec{x}, \vec{a})$$

where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the position vector of an arbitrary point (x, y), $\vec{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ is the position vector of A, $Hf(p, q)$ is the Hessian of f at A, and $R_2(\vec{x}, \vec{a})$ is the remainder term which satisfies

$$\frac{|R_2(\vec{x}, \vec{a})|}{|\vec{x} - \vec{a}|^2} \to 0 \qquad \text{as} \qquad \vec{x} \to \vec{a}.$$

The dot, $\cdot$, is the usual dot product of vectors. If we write this out, we have

$$\begin{aligned} f(x,y) &= f(p,q) + \begin{pmatrix} f_x(p,q) \\ f_y(p,q) \end{pmatrix} \cdot \begin{pmatrix} x-p \\ y-q \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-p \\ y-q \end{pmatrix} \cdot \left[\begin{pmatrix} f_{xx}(p,q) & f_{xy}(p,q) \\ f_{yx}(p,q) & f_{yy}(p,q) \end{pmatrix} \begin{pmatrix} x-p \\ y-q \end{pmatrix} \right] \\ &= f(p,q) + \begin{pmatrix} f_x(p,q) \\ f_y(p,q) \end{pmatrix} \cdot \begin{pmatrix} x-p \\ y-q \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-p \\ y-q \end{pmatrix} \cdot \begin{pmatrix} f_{xx}(p,q)(x-p) + f_{xy}(p,q)(y-q) \\ f_{xy}(p,q)(x-p) + f_{yy}(p,q)(y-q) \end{pmatrix} \\ &= f(p,q) + f_x(p,q)(x-p) + f_y(p,q)(y-q) \\ &\quad + \frac{1}{2}\left[f_{xx}(p,q)(x-p)^2 + 2f_{xy}(p,q)(x-p)(y-q) + f_{yy}(p,q)(y-q)^2 \right] \end{aligned}$$

Just as in Calc II, this series has a radius of convergence R which it is valid in (it gives a disk of points about the point A of radius R which the series is valid in).

With this, we can quantify the change in f between the point (p, q) and some point (x, y), which is given by

$$\Delta f = f(x,y) - f(p,q)$$

as

$$\Delta f = f_x(p,q)(x-p) + f_y(p,q)(y-q) + \frac{1}{2}\left[f_{xx}(p,q)(x-p)^2 + 2f_{xy}(p,q)(x-p)(y-q) + f_{yy}(p,q)(y-q)^2 \right]$$

2.3. A brief lemma. To prove the second derivative test, we use the following lemma:

Lemma. *Consider the quadratic ($A \neq 0$) function $g(x) = Ax^2 + 2Bx + C$.*

1. *If $AC - B^2 > 0$, and $A > 0$ or $C > 0$, then $g(x) > 0$ for all x.*
2. *If $AC - B^2 > 0$, and $A < 0$ or $C < 0$, then $g(x) < 0$ for all x.*
3. *If $AC - B^2 < 0$, then there are x values such that $g(x) > 0$ and some x values with $g(x) < 0$.*

Proof. We prove these on a case by case basis

(1) Suppose we have that $AC - B^2 > 0$. If $A > 0$, then $\lim_{x\to\infty} g(x) = \infty$, meaning that $g(x) > 0$ for some x. On the other hand, if $C > 0$ then $g(0) > 0$, so again, we know there are x where $g(x) > 0$. If g ever becomes negative, then by the intermediate value theorem, we know that g has zeros. We can use the quadratic formula to search for the x values for which $g(x) = 0$:

$$x = \frac{-2B \pm \sqrt{(2B)^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

Since $AC - B^2 > 0$, this means that $B^2 - AC < 0$, and so the x values from the quadratic formula above are not real (they have a nonzero imaginary part). This means that $g(x)$ is never zero for any x, and so g never crosses below the x-axis, hence $g(x) > 0$ for all z.

(2) Almost identically to the previous part, suppose we have that $AC - B^2 > 0$. If $A < 0$, then $\lim_{x\to\infty} g(x) = -\infty$, meaning that $g(x) < 0$ for some x. On the other hand, if $C < 0$ then $g(0) < 0$, so again, we know there are x where $g(x) < 0$. If g ever becomes positive, then by the intermediate value theorem, we know that g has zeros. We can use the quadratic formula to search for the x values for which $g(x) = 0$:

$$z = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

Since $AC - B^2 > 0$, this means that $B^2 - AC < 0$, and so the x values from the quadratic formula above are not real (they have a nonzero imaginary part). This means that $g(x)$ is never zero for any x, and so g never crosses above the x-axis, hence $g(x) < 0$ for all z.

(3) Now, the fun part! Assume that $AC - B^2 < 0$. This means that $B^2 - AC > 0$. Let's search for when $g(x) = 0$. It is zero for the following z values:

$$x = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

Since $B^2 - AC > 0$, this means there are two real values of x for which $g(x)$ is zero. Since $g(x)$ has exactly two zeroes, it crosses the x-axis exactly twice. This must mean that $g(x)$ takes on both negative and positive values. (If you're having trouble with this, just draw a few pictures of parabolas which intersect the x-axis twice to see it.)

$\square$

2.4. **Proof of the test.** Let us recall the theorem that we want to prove

Theorem (Second Partials Test)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that (a, b) is a critical point of f, i.e., $\nabla f(a, b) = \vec{0}$. Let*

$$D(a, b) = \det(Hf(a, b)),$$

then

(1) *If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
(2) *If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
(3) *If $D(a, b) < 0$, then (a, b) is a saddle point.*

Recall that a point (a, b) is a

(1) *local minimum* if $\Delta f \geq 0$ for all (x, y) near (a, b),
(2) *local maximum* if $\Delta f \leq 0$ for all (x, y) near (a, b),
(3) *saddle point* if $\Delta f > 0$ for some (x, y) near (a, b) and $\Delta f < 0$ for other (x, y) near (a, b).

Alright! Here we go!

Proof. Since (a,b) is a critical point, we know that $f_x(a,b) = f_y(a,b) = 0$, and so

$$\begin{aligned}\Delta f &= \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] \\ &= \frac{(y-b)^2}{2}\left[f_{xx}(a,b)\left(\frac{x-a}{y-b}\right)^2 + 2f_{xy}(a,b)\left(\frac{x-a}{y-b}\right) + f_{yy}(a,b)\right]\end{aligned}$$

Obviously, we avoid picking points where $y = b$ for this proof, otherwise this will not work (we can pick points with y-coordinate as close to b as we want though). If we let $z = \dfrac{x-z}{y-b}$, then we have

$$\Delta f = \frac{(y-b)^2}{2} g(z)$$

where $g(z) = f_{xx}(a,b)z^2 + 2f_{xy}(a,b)z + f_{yy}(a,b)$. Since $\dfrac{(y-b)^2}{2} \geq 0$, the sign of Δf is completely determined by $g(z)$. But $g(z)$ is exactly of the form in the lemma above, where

$$\begin{cases} A &= f_{xx}(a,b) \\ B &= f_{xy}(a,b) \\ C &= f_{yy}(a,b) \\ AC - B^2 &= \det(Hf(a,b)) = D(a,b) \end{cases}$$

(1) Suppose that $D(a,b) > 0$ and $f_{xx}(a,b) > 0$. This means that $AC - B^2 > 0$ and $A > 0$ in terms of the lemma. In this case it means that $g(z) > 0$ for all z. Thus Δf is always positive, meaning that (a,b) is a local minimum.
(2) Suppose that $D(a,b) > 0$ and $f_{xx}(a,b) < 0$. This means that $AC - B^2 > 0$ and $A < 0$ in terms of the lemma. In this case it means that $g(z) < 0$ for all z. Thus Δf is always negative, meaning that (a,b) is a local maximum.
(3) Suppose that $D(a,b) < 0$. Then $AC - B^2 < 0$ and so $g(z)$ is positive for some z, and negative for others. This means that Δf is positive for some points (x,y) and negative for others. Thus (a,b) is a saddle point.

A brief caveat for (1) and (2): Technically we have not shown that $\Delta f \geq 0$ (resp. $\Delta f \leq 0$) for points (x,y) when $y = b$. To do this, in the equation for Δf, instead of factoring out $(y-b)^2$, we instead factor out $(x-a)^2$. In this case $g(w) = f_{yy}(a,b)w^2 + 2f_{xy}(a,b)w + f_{xx}(a,b)$, where $w = \dfrac{y-b}{x-a}$. This is why the conditions on C are in the lemma! This allows us to use the points when $y = b$ (and disallows when $x = a$, but this was already taken care of in the previous case). $\square$

There is also the part that if $D(a,b) = 0$, then the test fails. This has to do with the nature of matrices. If the determinant of a matrix is zero, then that matrix is called *degenerate*. A degenerate matrix "maps one or more directions to zero". This usually corresponds to something where you have a whole line of critical points, e.g., the function $f(x,y) = -(x-y)^2$, or something more subtle, e.g., $f(x,y) = x^3 + y^3$.

References

[1] Susan J. Colley, *Vector Calculus, 4e*. Pearson Education, Inc. Boston. 2012.